CHAPTER 7 SEQUENTIAL INVESTMENT OPTIONS

Thus far, it has been assumed that the investment amount is paid instantaneously upon exercise of the real option, that is when commencing the investment. Often, investment opportunities require a sequence of expenditures, so that interim "mini-investments" are necessary over a time line in order to keep the ultimate investment opportunity option alive. This chapter allows for sequential investment options (also termed instalment options), where it is assumed that interim expenditures are completely sunk costs, having no alternative or scrap value.

Assume first the investment program involves required initial expenditures (the real option premium), a second phase of required investment expenditures (D), and a final development phase, when then the project values (V) are realized. The essential aspect of this characterized program is that managers have a choice of whether to pay the interim expenditure, and then the development cost (K). This program constitutes a call option on a further call option. If all costs are considered "sunk costs", the initial expense at t_0 is an irrecoverable premium for a call option to pay D at t_1, which is itself a premium for an option to pay K at t_2, to receive then the project values. Without management flexibility not to pay D or K, perhaps such a program should be valued using present values. With management discretion, real option models are appropriate since future expenditures can be cancelled. The first stage decisions are based on the difference between perceived value (including future options) and cost at or before exercise dates. The transitions between the stages are sequential options.

These models are suitable for any investment program, where there are required interim expenditures for program continuance such as: (a) a telecommunications company contemplating providing intermediate services and looking to maintain or increase line usage, or a mobile operator initially bidding for a 4G license, that requires R&D at a first stage, and then implementation expenditures; (b) an E_Commerce software or a search service provider, which aims to add advertising, and then content in sequences, each requiring R&D and marketing expenditures; and (c) a property developer, who pays an initial price for development land, where there are required interim decontamination expenses, and then final construction costs.

Here are three real option valuation methods, starting with a simple European compound option, extended to a European compound exchange option. Finally, an American perpetual exchange option is presented, allowing for several stages of investment expenditures (and critical values which justify making those expenditures).

The simplest European sequential model is the Geske (1979) compound call on a call option. The simple European exchange option is an adapted Margrabe (1978) exchange option, set in a compound option format. This assumes that both the development costs and the ultimate project value are both stochastic, and costs (D) must be spent at t_1 in order to keep alive the option to exchange K for V at $T(=t_2)$.

Building on Adkins and Paxson (2013), a multi-stage sequential American perpetual exchange model is provided.

7.1 SEQUENTIAL EUROPEAN REAL OPTIONS

Geske (1979) developed an analytic framework for a European option, where in order to keep the option alive an interim expenditure is required. There is a critical value V* which justifies making the interim expenditure. Assume that developed values (V) follow a geometric Brownian motion:

$$dV = (\mu_V - \delta_V)Vdt + \sigma_V Vdz_v \tag{7.1}$$

where μ_V is the equilibrium expected drift rate, δ_V is the income rate (or payout rate) of V, and σ_V is the volatility. Let the value of a call on a call be the real option value C_c, where D is the interim expenditure required at time τ'=.5 (or another fraction), and K the investment cost at time τ. The value of a call on a call C_c is given by

$$C_c = Ve^{-\delta\tau} B(a_1, d_1; \rho) - Ke^{-r(\tau-\tau')} B(a_2, d_2; \rho) - De^{-r\tau'} N(a_2) \quad (7.2)$$

where ρ is the correlation coefficient between the overlapping Brownian motion increments, which is defined as $\rho = \sqrt{\tau'/\tau}$, and N(.) and B(.) are the standard cumulative univariate and bivariate normal distributions with parameters:

$$a_1 = \frac{\ln(V/V^*) + (r - \delta + 0.5\sigma^2)\tau'}{\sigma\sqrt{\tau'}},\ a_2 = a_1 - \sigma\sqrt{\tau'}, \quad (7.3)$$

$$d_1 = \frac{\ln(V/K) + (r - \delta + 0.5\sigma^2)\tau}{\sigma\sqrt{\tau}},\ d_2 = d_1 - \sigma\sqrt{\tau}.$$

The critical price, V^*, is obtained by solving the value matching condition:

$$V^* e^{-\delta(\tau-\tau')} N(d_1^*) - Ke^{-r(\tau-\tau')} N(d_2^*) = D \quad (7.4)$$

$$d_1^* = \frac{\ln(V^*/K) + (r - \delta + 0.5\sigma^2)(\tau - \tau')}{\sigma\sqrt{\tau - \tau'}},\ d_2^* = d_1^* - \sigma\sqrt{\tau - \tau'} \quad (7.5)$$

Using standard parameters, the Geske European sequential investment model is shown in Figure 7.1. Use Tools/Solver to solve equation (7.4)-D=0. In column B, V* is 95, almost 100 the current value, for this is a more or less at the money compound call option. If V is 100 at time τ', the payment D=20 should be made in order to keep the ultimate call option alive.

Since the Geske compound option model is European, it is at best a first estimate for long-lived sequential options. As also shown in Figure 7.1 column C, the compound option value increases as the time to ultimate exercise increases (with D at the half way time).

Figure 7.1

	A	B	C	D
1	**GESKE EUROPEAN COMPOUND CALL OPTION**			
2				
3	DEVELOPMENT TIME τ'	1	3.00	
4	INVESTMENT TIME τ	2	3.75	
5	INTEREST RATE	0.04	0.04	
6	V YIELD	0.04	0.04	
7	VALUE VOLATILITY	0.20	0.20	
8	V	100.00	100.00	
9	D	10.00	10.00	
10	K	90.00	90.00	
11	**REAL OPTION VALUE**	**7.9079**	**11.9899**	B8*EXP(-B6*B4)*B26-B10*EXP(-B5*B4)*B27-B9*EXP(-B5*B3)*B25
12	τ-τ'	1.00	0.75	B4-B3
13	V*	95.42	96.65	
14	d1	0.39	0.50	(LN(B13/B10)+((B5-B6+0.5*B7^2)*B12))/(B7*SQRT(B12))
15	d2	0.19	0.33	B14-B7*SQRT(B12)
16	N1	0.65	0.69	NORMSDIST(B14)
17	N2	0.58	0.63	NORMSDIST(B15)
18	EQ 7.4	10.00	10.00	B13*EXP(-B6*B12)*B16-B10*EXP(-B5*B12)*B17
19	EQ 7.4-D	0.00	0.00	B18-B9
20	ρ(τ'/τ)	0.71	0.89	SQRT(B3/B4)
21	d1,t1	0.33	0.27	(LN(B8/B13)+(B5-B6+0.5*B7^2)*B3)/(B7*SQRT(B3))
22	d1,t2	0.51	0.47	(LN(B8/B10)+(B5-B6+0.5*B7^2)*B4)/(B7*SQRT(B4))
23	d2,t1	0.13	-0.07	B21-B7*SQRT(B3)
24	d2,t2	0.23	0.08	B22-B7*SQRT(B4)
25	N2	0.55	0.47	NORMSDIST(B23)
26	B1	0.55	0.57	BiVariateNormalCDF(B21,B22,B20)
27	B2	0.45	0.42	BiVariateNormalCDF(B23,B24,B20)
28				
29	**The first five inputs are the D and K timing estimates, the interest rate, and the**			
30	**value yield and volatility.**			
31	**The next three inputs are V, D and K estimates.**			
32	**Real call option value assumes V* is the value above which D should be paid at τ'.**			
33	**USE TOOLS/SOLVER, SETTING B19=0 BY CHANGING B13.**			

7.2 SEQUENTIAL EUROPEAN EXCHANGE OPTION

It is easy to extend this compound option model to a European sequential exchange real option. Suppose that the D and development costs K follow a diffusion process similar to that for V:

$$dK = (\mu_K - \delta_K)Kdt + \sigma_K Kdz_K \tag{7.6}$$

where μ_K is the drift term (the expected cost escalation), δ_K is the payout rate on similar investment cost businesses, σ_K is the volatility of the investment cost, and the correlation between the Wiener processes is ρ. Assuming that the exercise price of the first (compound) option, D, is expressed as a fixed proportion (Q%) of K, i.e., D=QK, Carr (1988) gives the solution for the European compound exchange call option:

$$w_C(V,K,D,\tau,\tau') = Ve^{-\delta_V \tau} B\left(a_1, b_1; \sqrt{\frac{\tau'}{\tau}}\right)$$

$$- Ke^{-\delta_K \tau} B\left(a_2, b_2; \sqrt{\frac{\tau'}{\tau}}\right) - De^{-\delta_K \tau'} N(a_2) \tag{7.7}$$

where

$$a_1 = \frac{\ln(X/X^*) + (\delta_K - \delta_V + 0.5\sigma^2)\tau'}{\sigma\sqrt{\tau'}}, \ \mathrm{a}_2 = \mathrm{a}_1 - \sigma\sqrt{\tau'}, \tag{7.8}$$

$$b_1 = \frac{\ln(X) + (\delta_K - \delta_V + 0.5\sigma^2)\tau}{\sigma\sqrt{\tau}}, \ \mathrm{b}_2 = \mathrm{b}_1 - \sigma\sqrt{\tau}, \tag{7.9}$$

$$X = \frac{V}{K}, \ \sigma = \sqrt{\sigma_V^2 - 2\rho\sigma_V\sigma_K + \sigma_K^2}.$$

N(.) and B(, ;) are the standard normal cumulative univariate and bivariate distributions. At time τ', one would exercise the compound call and obtain the underlying European exchange call option if the critical price ratio is such that $X^* < X_{\tau'}$. The critical price ratio, X^*, above which the compound option should be exercised at time τ' can be obtained using the solution for the European exchange call option:

$$X^* e^{(\delta_K - \delta_V)(\tau - \tau')} N(b_1^*) - N(b_2^*) = Q \tag{7.10}$$

where

$$b_1^* = \frac{\ln\left(X^*\right) + \left(\delta_K - \delta_V + 0.5\sigma^2\right)\left(\tau - \tau'\right)}{\sigma\sqrt{\tau - \tau'}}, \ \mathrm{b}_2^* = \mathrm{b}_1^* - \sigma\sqrt{\tau - \tau'} \tag{7.11}$$

Figure 7.2

	A	B	C
1	**EUROPEAN COMPOUND EXCHANGE OPTION**		
2			
3	DEVELOPMENT TIME τ'	1	
4	INVESTMENT TIME τ	2	
5	V YIELD	0.04	
6	K YIELD	0.04	
7	VALUE VOLATILITY	0.20	
8	K VOLATILITY	0.20	
9	CORRELATION	0.50	
10	EXCHANGE VOLATILITY	0.20	SQRT(B7^2+B8^2-2*B9*B7*B8)
11	V	100.00	
12	D	10.00	
13	K	90.00	
14	X=V/K	1.1111	B11/B13
15	**REAL OPTION VALUE**	**7.9075**	B11*EXP(-B5*B4)*B30-B13*EXP(-B6*B4)*B31-B12*EXP(-B6*B3)*B29
16	τ-τ'	1.00	B4-B3
17	X*	1.06	
18	d1	0.63	(LN(B11/B13)+((B5-B6+0.5*B10^2)*B16))/(B10*SQRT(B16))
19	d2	0.43	B18-B10*SQRT(B16)
20	N1	0.73	NORMSDIST(B18)
21	N2	0.67	NORMSDIST(B19)
22	EQ 7.10	0.11	B17*EXP(-B5*B16)*B20-EXP(-B6*B16)*B21
23	EQ 7.10-D/K	0.00	B22-(B12/B13)
24	ρ(τ'/τ)	0.71	SQRT(B3/B4)
25	d1,t1	0.32	(LN(B14/B17)+(B6-B5+0.5*B10^2)*B3)/(B10*SQRT(B3))
26	d1,t2	0.51	(LN(B14)+(B6-B5+0.5*B10^2)*B4)/(B10*SQRT(B4))
27	d2,t1	0.12	B25-B7*SQRT(B3)
28	d2,t2	0.23	B26-B7*SQRT(B4)
29	N2	0.55	NORMSDIST(B27)
30	B1	0.54	BiVariateNormalCDF(B25,B26,B24)
31	B2	0.45	BiVariateNormalCDF(B27,B28,B24)
32			
33	**The first five inputs are the D and K timing estimates, the**		
34	**value and cost yields and volatilities, and correlation.**		
35	**After calculating the exchange volatility, the next three inputs are V, D and K estimates.**		
36	**Real call option value assumes X* is the value above which D should be paid at τ'.**		
37	**USE TOOLS/SOLVER, SETTING B23=0 BY CHANGING B17.**		

The European sequential option model assumes that D cannot occur until τ' and K is only paid or exercised at τ. This is mechanical, and does not allow management any flexibility, except to choose whether to make the investment decisions. The input parameters are chosen so that the ROV is the same as in the previous figure. Different inputs for K yield and volatility, and correlation, will yield different results.

7.3 AMERICAN SEQUENTIAL MULTI-STAGE EXCHANGE REAL OPTIONS

This section provides a model which can easily be extended to cover a multiple stage sequential investment opportunity. The analytical solution depends on assuming a probability of catastrophic failure at each investment stage that declines as the project nears completion, which is a characteristic of many R&D, exploration and infrastructure projects. The project can then be interpreted as a collection of investment stages, such that no stage investment, except the first, can be started until the preceding stage has been completed. Success at each stage is not guaranteed because of the possibility of a catastrophic failure that reduces the option value to zero. The project value is realized when all the stages have been successfully completed. A typical four-stage opportunity involves: (i) undertaking basic research. (ii) developing a marketable product, (iii) testing its viability and (iv) implementing the infrastructure for launch and delivery. Multiple sequential investment opportunities are common amongst industries as diverse as oil exploration and mining, aircraft manufacture, pharmaceuticals and consumer electronics.

Schwartz and Moon (2000) model a new drug development process which consists of four distinct phases, each with a positive probability of failure, although not necessarily declining over time. Cortazar, Schwartz and Casassus (2003) describe four natural resource exploration stages of a project with technical success probability increasing over each phase, and then a production phase which is subject to commodity price uncertainty. Pennings and Sereno (2011) study the development path of a new medicine over seven phases, with a probability of failure declining over time.

Making an investment at a stage depends on whether the prevailing project value is of sufficient magnitude to economically justify committing the investment cost, or whether it is more desirable to wait for more favorable conditions. There are three sources of uncertainty, the stochastic project value and the investment cost, and the probability of a catastrophic failure, which are considered in a closed-form rule for the investment decision at each of the project stages.

Other authors simplify the multiple investment stage problems for obtaining a meaningful solution. Building on the valuation of sequential exchange opportunities by Carr (1988), Lee and Paxson (2001) use an element of European style compound options (and approximation of an American option phase) for formulating a two-stage sequential investment. Brach and Paxson (2001) examine a two-stage sequential investment opportunity similar to the formulation currently under study but they confine their attention more to valuation. Childs and Triantis (1999) formulate a multiple sequential investment model with interaction and obtain a solution through using a trinomial lattice. For all of these expositions, the solution is either not analytical or is restricted to only two stages.

Cassimon et al. (2004) study American-type investment options, but provide a solution based on the complex multivariate distribution available in some mathematical programmes. Building on Adkins and Paxson (2011), Adkins and Paxson (2013) suggest an analytic solution for N-stage sequential investments.

Consider an investment project made up of a discrete number of sequential stages, each involving an individual non-zero investment cost. The project as an entity is not fully implemented and the project value not realized until all of the sequential stages have been successfully completed. Each successive investment stage relies on the successful completion of the investment made at the preceding stage, but the stage timing is not specified. Each investment stage is ordered by the number J of remaining stages, including the current one, until project completion. The decision making position is first examined for the ultimate stage where $J=1$, and then by replication for the preceding stages, incrementally. At the ultimate stage, the decision whether or not to make an investment in a real asset is decided by whether or not the option value at $J=1$ fully compensates the expected net present value of the cash flow stream rendered by the asset. At the penultimate stage $J=2$, whether to make an expenditure to obtain the investment option at $J=1$ depends on whether or not the option value at $J=2$ fully compensates for the net option value at $J=1$. This procedure is then replicated incrementally for stages greater than 2.

A representation of the sequential investments process for a $J=N$ stage project is illustrated in Figure 7.3. This figure reveals the ordered sequence of stage investments comprising the project. It also shows that after an investment, the possible outcomes are success and failure. If all the stage outcomes are successful, then the entire project is successfully completed and its value can be realized. However, there is a possibility of failure at each stage. Although the investment is committed, the stage may not be successfully completed owing to fundamental irresolvable technical or market impediments, in which case, the option value instantly falls to zero and the project is abandoned without any value. The probability of failure at stage J is denoted by λ_J where $0 \le \lambda_J < 1 \, \forall J$.

Situations where an investment can produce an innovative breakthrough and generate an unanticipated increase in the project value are ignored. Also, other forms of optionality, such as terminating a project before completion for its abandonment value, are not considered.

The value of the project is defined by V. The investment expenditure made at any stage J is denoted by K_J for all possible values of J. Both the project value and the set of investment expenditures are treated as stochastic. It is assumed that they are individually well described by the geometric Brownian motion process:

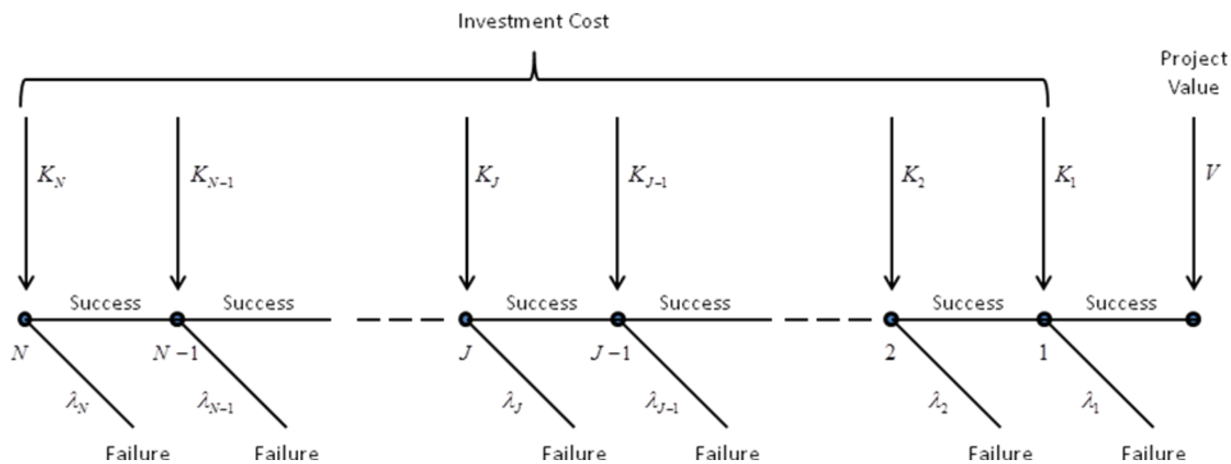
$$\mathrm{d}X = \alpha_X X \mathrm{d}t + \sigma_X X \mathrm{d}z_X \tag{7.12}$$

for $X \in \{V, K_J \, \forall J\}$, where α_X represent the respective drift parameters, σ_X the respective instantaneous volatility parameter, and $\mathrm{d}z_X$ the respective increment of a standard Wiener process. Dependence between any two of the factors is represented by the covariance term; so, for example, the covariance between the real asset value and the investment expenditure at stage J is specified by:

$$\mathrm{Cov}\left[\mathrm{d}V, \mathrm{d}K_J\right] = \rho_{VK_J} \sigma_V \sigma_{K_J} \mathrm{d}t .$$

Figure 7.3

Sequential Investment Process



Different stages may have different factor volatilities and correlations. The risk-free rate is r, and the investment expenditure at each stage K is assumed to be instantaneous.

One-Stage Model

The stage $J = 1$ model represents the investment opportunity for developing a project value V following the investment cost K_1, given that the research effort may fail totally with probability λ_1. The value F_1 of the investment opportunity at stage $J = 1$ depends on the project value and the investment cost, so $F_1 = F_1\left(V, K_1\right)$. By Ito's lemma, the risk neutral valuation relationship is:

$$\tfrac{1}{2}\sigma_V^2 V^2 \frac{\partial^2 F_1}{\partial V^2} + \tfrac{1}{2}\sigma_1^2 K^2 \frac{\partial^2 F_1}{\partial K_1^2} + \rho_{VK_1}\sigma_V\sigma_{K_1} V K_1 \frac{\partial^2 F_1}{\partial V \partial K_1}$$
$$+ \theta_V V \frac{\partial F_1}{\partial V} + \theta_{K_1} K_1 \frac{\partial F_1}{\partial K_1} - \left(r + \lambda_1\right) F_1 = 0, \tag{7.13}$$

where the θ_X for $X \in \{V, K_J \,\forall J\}$ denote the respective risk neutral drift rate parameters. The generic solution is the two-factor power function:

$$F_1 = A_1 V^{\beta_1} K_1^{\eta_1}, \tag{7.14}$$

where β_1 and η_1 denote the generic unknown parameters for the two factors, project value and investment cost, and A_1 denotes a generic unknown coefficient. Since the option value is always non-negative, $A_1 \geq 0$. We conjecture that $\beta_1 \geq 0$ and $\eta_1 < 0$, and $\beta_1 + \eta_1 = 1$. The power parameter values satisfy the characteristic root function:

$$Q_1\left(\beta_1, 1-\beta_1\right) = \tfrac{1}{2}\sigma_1^2\beta_{12}\left(\beta_1 - 1\right) + \beta_1\left(\theta_V - \theta_{K_1}\right) - \left(r + \lambda_1 - \theta_{K_1}\right) = 0, \tag{7.15}$$

where $\sigma_1^2 = \sigma_V^2 + \sigma_{K_1}^2 - 2\rho_{V,K_1}\sigma_V\sigma_{K_1}$. (7.16)

The threshold levels for the project value and the investment cost signaling the optimal exercise for the investment option at stage $J = 1$ are denoted by $\hat{V}_1$ and $\hat{K}_1$, respectively. The value matching relationship describes the conservation equality at optimality that the option value $\hat{F}_1 = F_1\left(\hat{V}_1, \hat{K}_1\right)$ exactly compensates the net asset value $\hat{V}_1 - \hat{K}_1$. Then:

$$A_1 \hat{V}_1^{\beta_1} \hat{K}_1^{\eta_1} = \hat{V}_1 - \hat{K}_1. \tag{7.17}$$

There are two associated smooth pasting conditions, one for each factor, which can be expressed as:

$$A_1 \hat{V}_1^{\beta_1} \hat{K}_1^{\eta_1} = \frac{\hat{V}_1}{\beta_1}. \tag{7.18}$$

$$A_1 \hat{V}_1^{\beta_1} \hat{K}_1^{\eta_1} = -\frac{\hat{K}_1}{\eta_1} \tag{7.19}$$

Further, the threshold levels are related by:

$$\hat{V}_1 = \frac{\beta_1}{\beta_1 - 1}\hat{K}_1, \tag{7.20}$$

with $$A_1 = \beta_1^{-\beta_1}\left(\beta_1 - 1\right)^{\beta_1 - 1}. \tag{7.21}$$

Two-Stage Model

At the preceding stage, $J = 2$, the viability of committing an investment K_2 to acquire the option to invest F_1 is compared to the value of the compound option F_2 with the net benefits $F_1 - K_2$. F_2 depends on the three factors V, K_1 and K_2, so $F_2 = F_2\left(V, K_1, K_2\right)$. By Ito's lemma, the risk neutral valuation relationship for F_2 is:

$$\begin{aligned}
&\tfrac{1}{2}\sigma_V^2 V^2 \frac{\partial^2 F_2}{\partial V^2} + \tfrac{1}{2}\sigma_{K_1}^2 K_1^2 \frac{\partial^2 F_2}{\partial K_1^2} + \tfrac{1}{2}\sigma_{K_2}^2 K_2^2 \frac{\partial^2 F_2}{\partial K_2^2} \\
&+\rho_{V,K_1}\sigma_V\sigma_{K_1} V K_1 \frac{\partial^2 F_2}{\partial V \partial K_1} + \rho_{V,K_2}\sigma_V\sigma_{K_2} V K_2 \frac{\partial^2 F_2}{\partial V \partial K_2} + \rho_{K_1,K_2}\sigma_{K_1}\sigma_{K_2} K_1 K_2 \frac{\partial^2 F_2}{\partial K_1 \partial K_2} \\
&+\theta_V V \frac{\partial F_2}{\partial V} + \theta_{K_2} K_2 \frac{\partial F_2}{\partial K_2} + \theta_{K_1} K_1 \frac{\partial F_2}{\partial K_1} - \left(r + \lambda_2\right) F_2 = 0.
\end{aligned} \tag{7.22}$$

The solution to (7.22) is a product power function, with generic form:

$$F_2 = A_2 V^{\beta_{24}} K_1^{\eta_{21}} K_2^{\eta_{22}} = B_2 [F_1]^{\phi_2} K_2^{1-\phi_2} \tag{7.23}$$

where β_2, η_{21} and η_{22} denote the generic unknown parameters for the three factors, project value and investment expenditure at stage-one and -two respectively, and A_2 denotes an unknown coefficient. $B_2 = (\phi_2 - 1)^{(\phi_2 - 1)} / \phi_2^{\phi_2}$ (7.24)

The stage-two threshold levels signaling an optimal exercise are represented by $\hat{V}_2$, $\hat{K}_1$ and $\hat{K}_2$ for V, K_1 and K_2, respectively. The set $\left\{\hat{V}_2, \hat{K}_1, \hat{K}_2\right\}$ forms the boundary that discriminates between the "exercise" decision and the "wait" decision. The equilibrium amongst the threshold levels is the value matching relation that is expressed as:

$$A_2\hat{V}_2^{\beta_2}\hat{K}_1^{\eta_{21}}\hat{K}_2^{\eta_{22}} = A_1\hat{V}_2^{\beta_1}\hat{K}_1^{1-\beta_1} - \hat{K}_2, \quad (7.25)$$

where A_1 and β_1 are known from the evaluation for stage-one. There are three smooth pasting conditions, one for each of the three factors V, K_1 and K_2, respectively, can be expressed as:

$$\beta_2 A_2\hat{V}_2^{\beta_2}\hat{K}_1^{\eta_{21}}\hat{K}_2^{\eta_{22}} = \beta_1 A_1\hat{V}_2^{\beta_1}\hat{K}_1^{1-\beta_1}, \quad (7.26)$$

$$\eta_{21} A_2\hat{V}_2^{\beta_2}\hat{K}_1^{\eta_{21}}\hat{K}_2^{\eta_{22}} = (1-\beta_1) A_1\hat{V}_2^{\beta_1}\hat{K}_1^{1-\beta_1}, \quad (7.27)$$

$$\eta_{22} A_2\hat{V}_2^{\beta_2}\hat{K}_1^{\eta_{21}}\hat{K}_2^{\eta_{22}} = -\hat{K}_2. \quad (7.28)$$

As a simplification in calculating the solution values, let $\phi_2 = \beta_2 / \beta_1 \geq 0$, then by using the substitutions $\beta_2 = \phi_2\beta_1$, $\eta_{21} = (1-\beta_1)\phi_2$ and $\eta_{22} = 1-\phi_2$, the quadratic function Q_2 can be expressed as:

$$Q_2 = \tfrac{1}{2}\phi_2(\phi_2 - 1)\sigma_2^2 + \phi_2\left(r + \lambda_1 - \theta_{K_2}\right) - \left(r + \lambda_2 - \theta_{K_2}\right) = 0. \quad (7.29)$$

where

$$\begin{aligned}\sigma_2^2 &= \beta_1^2\sigma_V^2 + (1-\beta_1)^2\sigma_{K_1}^2 + \sigma_{K_2}^2 \\ &\quad + 2\beta_1(1-\beta_1)\rho_{VK_1}\sigma_V\sigma_{K_1} - 2\beta_1\rho_{VK_2}\sigma_V\sigma_{K_2} - 2(1-\beta_{12})\rho_{K_1K_2}\sigma_{K_1}\sigma_{K_2}.\end{aligned} \quad (7.30)$$

The value of ϕ_{24} is evaluated as the positive root of Q_2.

$$\hat{V}_2 = \frac{\beta_1}{\beta_1 - 1}\left\{\frac{\phi_2(\beta_1 - 1)}{\phi_2 - 1}\right\}^{\frac{1}{\beta_1}} \hat{K}_1^{\frac{\beta_1 - 1}{\beta_1}} \hat{K}_2^{\frac{1}{\beta_1}} \quad (7.31)$$

Numerical Illustrations

Figure 7.4 is a spreadsheet evaluation on an illustration involving a 2-stage sequential investment project, solving two sets of simultaneous equations, EQs 7.15, 7.17, 7.18 and 7.19 for the first stage, and 7.25-7.29 for the second stage. The set of probabilities of catastrophic failure at the stages adheres to the condition $\lambda_1 < \lambda_2$. Initially, the variances for the investment costs at the two stages have been set to be equal, the covariance terms between the four factors to equal zero, and the K thresholds are all assumed to be the same as the current value, so the threshold justifying investment at each stage is the ratio of $\hat{V}$ to the nominal investment costs remaining.

Figure 7.4

	A	B	C	D	E	F	G	H	I	J	K
1	**STAGE ONE**		**STAGE TWO**								
2	INPUT			INPUT							
3	V	100.00	100.00	V	**STAGE**	**VOLATILITY**	**ϕ**		**V^**	**V^-ΣK$_N$**	
4	K1	90.00	90.00	K1	1	0.2000	3.0000		**135.0000**	45.0000	
5	K2	10.00	10.00	K2	2	0.4472	1.3660		**126.8369**	26.8369	
6	σV	0.20	0.20	σV							
7	σK1	0.20	0.20	σK1	**STAGE**	**ROV**	**A**	**β**	**η₁**	**η₂**	
8	σK2	0.20	0.20	σK2	1	**18.2899**	0.1481	3.0000	-2.0000		
9	ρ VK1	0.50	0.50	ρ VK1	2	**10.3129**	0.0333	4.0981	-2.7321	-0.3660	
10	ρ VK2	0.50	0.50	ρ VK2							
11	ρ K1K2	0.00	0.00	ρ K1K2							
12	r	0.04	0.04	r							
13	θV	0.00	0.00	θV							
14	θK1	0.04	0.04	θK1							
15	θK2	0.04	0.04	θK2							
16	λ1	0.00	0.00	λ1							
17	λ2	0.05	0.05	λ2							
18	σ1^2	0.04	0.20	σ2^2							
19	Q(β,η)	0.0000	0.0000	Q(β,η)	0.5*(C18)*C34*(C34-1)+C34*(C12+C16-C15)-(C12+C17-C15)						
20	SP1	0.0000	0.0000	SP1	C27*C26*(C29^(C27-1))*(C31^C28)*(C32^C30)-B27*B26*(C29^(B27-1))*(C31^(1-B27))						
21	SP2	0.0000	0.0000	SP2	C28*C26*(C29^C27)*(C31^(C28-1))*(C32^C30)-(1-B27)*B26*(C29^B27)*(C31^(-B27))						
22	VM1	0.0000	0.0000	SP3	C30*C26*(C29^C27)*(C31^C28)*(C32^(C30-1))+1						
23			0.0000	VM1	C26*(C29^C27)*(C31^C28)*(C32^C30)-B26*(C29^B27)*(C31^(1-B27))+C32						
24	SOLVER: SET D25=0, CHANGING B26:C30										
25	SOLVER	0.0000	0.0000	0.00000							
26	A1	0.1481	0.0333	A2							
27	β1	3.00000	4.0981	β2							
28	η12	-2.00000	-2.7321	η2,1							
29	V1*	**135.00000**	**126.8369**	V2*							
30			-0.3660	η2,2							
31	K1*	**90.00000**	**90.0000**	K1*							
32			**10.0000**	K2*							
33	VOLATILITY	0.2000	0.4472	VOLATILITY							
34	β1	3.0000	1.3660	ϕ2							
35	ROV 1	18.2899	10.3129	ROV 2	IF(C3<C29,C26*(C3^C27)*(C4^C28)*(C5^C30),C3-C31-C32)						
36	Q(β,η)	0.5*(B18)*B27*(B27-1)+B27*(B13-B14)-(B12+B16-B14)									
37	SP1	B27*B26*(B29^(B27-1))*(B31^B28)-1									
38	SP2	B28*B26*(B29^B27)*(B31^(B28-1))+1									
39	VM1	B26*(B29^B27)*(B31^B28)-(B29-B31)									
40	ROV 1	IF(B3<B29,B26*(B3^B27)*(B4^B28),B27-B31)									
41	σ2^2	(B27^2)*(C6^2)+((1-B27)^2)*(C7^2)+(C8^2)+2*B27*(1-B27)*C9*C6*C7-2*B27*C10*C6*C8-2*(1-B27)*C11*C7*C8									

Figure 7.4 shows the results, using the backwardation principle so the $J=1$ stage is enumerated first, then the $J=2$ stage. The volatilities at each of the 2 stages, σ_1, and σ_2 are evaluated, as are the parameters ϕ_J for $J=1$. The volatilities at each

stage increase in magnitude as the stage in question becomes more distant from completion. As expected, the parameter values ϕ_J are all greater than one. Note that with these parameter values, $\hat{V}$ increases with the distance of the stage from completion, and with the stage volatility, as does the excess of the $\hat{V}$ over the assumed investment cost over each stage. The real option value (ROV), which is the option to continue the next stage if $V < \hat{V}$, and otherwise V less the remaining investment costs (or zero), decreases with the distance from the final state.

Figure 7.5 illustrates the matrix approach to solving the same problem as described in Adkins and Paxson (2013).

	A	B	C	D	E	F	G	H	I
1	**SEQUENTIAL MATRIX**	**2 STAGES**	**T**	**STAGE**	**VOLATILITY**	**ϕ**	**V^**	**V^-ΣK$_N$**	**ROV**
2	Project value V	100	3.75	1	0.2000	3.0000	135.0000	45.0000	**18.2899**
3	θ V	0	2.97	2	0.4472	1.3660	126.8368	26.8368	**10.3129**
4	σ V	**20%**		Input the correlations:					
5	**Stage 1**				**V**	K1	K2		Volatility
6	θ K1	**0.04**		V	**100%**	50%	50%		20%
7	σ K1	**20%**		K1	**50%**	100%	0%		20%
8	Failure probability: λ	**0.0000**		K2	**50%**	0%	100%		20%
9	**Stage 2**			Assumes all V, K correlations are the same as in E10, Ks are not correlated.					
10	θ K2	**0.0400**	T_2	(1/(B14-B3))*LN(G3/B2)/2					
11	σ K2	**0.2000**							
12	Failure probability: λ	**0.0500**							
13									
14	Risk-free rate	**0.0400**							
15	**Threshold Levels**								
16	K1^	**90**							
17	K2^	**10**							

The result is the same as in solving the two sets of equations. The expected investment timing at each stage is shown in C2 and C2, assuming a deterministic drift of V equal to the interest rate (and then for comparison with Geske, the result is divided by 2). This assumes some arbitrary process for the time that it takes V to reach V^, ignoring the fact that V is stochastic.

When the same Ts are used for the Geske compound European option over two stages, the ROV is 12, shown in Figure 1, (compared to 10.30 for A&P sequential American option over two stages). However, the comparison is between the Geske model with only one source of uncertainty, and the A&P model with several sources of uncertainty including the possibility of total project failure in the current stage.

There are many other alternative combinations of changes in value volatility, investment cost volatility at each stage, and probability of failure at each stage that could be simulated, to illustrate the power and surprises of viewing sequential investment opportunities (and eventually investment requirements over stages) using this model.

SUMMARY

Sequential investment options are appropriate when an investment program involves several stages, such as required initial expenditures (equivalent to a real option premium), a second phase of required investment expenditures (D), and a final development phase, when then the project values (V) are realized. The essential aspect of this characterized program is that managers have a choice about whether to pay the interim expenditure, and then the development cost (K).

This chapter presents three real option valuation methods, starting with a simple European compound option, extended to a European compound exchange option, and then an American perpetual exchange option, allowing for several stages of investment expenditures (and critical values which justify making those expenditures).

EXERCISES

EXERCISE 7.1

Roger Action wants to achieve his lifetime goal of monetizing a brilliant real option model on sequential investments, but realizes that there are two critical stages left requiring large investment sums. The final stage requires software and marketing costs in implementing a practical useful version of the model. The final stage is fairly simple with virtually no chance of failure, but both investment cost (=90) (increasing at 4% p.a.) and model value (=100) (increasing at 0%) are uncertain (volatility of 20%) but are not correlated. For the current stage, investment costs (=10) will increase at 4% p.a., are just as volatile and there is a 5% chance that the current stage will not succeed. The riskless interest rate is 4%, so β_1=3, and ϕ_2=1.366. Roger assumes that $\hat{K}_1 = 90$ $\hat{K}_2 = 10$, so he needs advice on the level of $\hat{V}_1$ and $\hat{V}_2$, and also the real option value at the current stage, since his wife wants him to sell this idea, and devote more time and effort to her.

The final stage model represents the investment opportunity for developing a project value V requiring the investment cost K_1. The real option value F_1 of the investment opportunity, depending on the project value and the investment cost, is:

$$F_1 = A_1 V^{\beta_1} K_1^{\eta_1}, \tag{1}$$

where β_1 and $\eta_1 = (1-\beta_1)$ are the power parameters for the two factors, and A_1 denotes an unknown coefficient. The threshold level which justifies making the investment is:

$$\hat{V}_1 = \frac{\beta_1}{\beta_1 - 1} \hat{K}_1, \tag{2}$$

with $$A_1 = \beta_1^{-\beta_1} \left(\beta_1 - 1\right)^{\beta_1 - 1}. \tag{3}$$

At the current stage the real option value is:

$$F_2 = B_2 [F_1]^{\phi_2} K_2^{1-\phi_2} \tag{4}$$

Let $$B_2 = (\phi_2 - 1)^{(\phi_2 - 1)} / \phi_2^{\phi_2}. \tag{5}$$

The value threshold which justifies commencing the stage 2 investment is:

$$\hat{V}_2 = \frac{\beta_1}{\beta_1 - 1} \left\{ \frac{\phi_2 \left(\beta_1 - 1\right)}{\phi_2 - 1} \right\}^{\frac{1}{\beta_1}} \hat{K}_1^{\frac{\beta_1}{\beta_1 - 1}} \hat{K}_2^{\frac{1}{\beta_1}} \tag{6}$$

PROBLEMS

PROBLEM 7.2 A bungalow in Putney has a restrictive covenant requiring the permission of the adjacent house owner in order to convert the bungalow into a modern house. That house owner is required to make extensive design and planning expenditures by the end of the next year prior to the construction of the new house. These expenditures and demolition costs are expected to be £150,000. A house of 3,000 square feet is envisioned, which currently would be worth £300 per square foot and costs £273 per square foot to build. The volatility of Putney houses is 20%, rental yield is 4% and interest rates are 4%. The redevelopment must occur at the end of five years. What is the value of this bungalow site? At what house value should the construction start?

PROBLEM 7.3 A bungalow in Putney has a restrictive covenant requiring the permission of the adjacent house owner in order to convert the bungalow into a modern house. That house owner is required to make extensive design and planning expenditures by the end of the next year prior to the construction of the new house. These expenditures and demolition costs are expected to be £150,000, and along with construction costs are 50% correlated with housing prices. A house of 3,000 square feet is envisioned, which currently would be worth £300 per square foot, and costs £273 per square foot to build. The volatility of Putney houses is 20%, the same as the construction costs, the "yield" on renting such a house is 4%, construction cost escalate by 4%, and interest rates are 4%. The redevelopment must occur at the end of five years. What is the value of this bungalow site? At what house value should the construction start?

PROBLEM 7.4

Willard Wang wants to enjoy the fruits of his research involving two expenditures (both equal to 50) K_1 at the end of the first year and K_2 at end of the second year. The current research price is 15, continuous cost is 10, the interest rate is 4% and the research yield is 4%. The research volatility is 20%. What's today's value of WW's research, and at what research price should he make the first and second investment expenditures?

PROBLEM 7.5

Pixit & Dindyck are planning a superior real options product PROD that will indicate optimal timing for perpetual multi-stage projects. They estimate that the current value of PROD is 81, costs 90 to make in three stages (10 in the current stage, 30 in the middle stage) has a volatility of 20%, interest rates are only 5%, while the yield on the PROD is expected to be 2%. The failure rate of the current stage is 10%, 5% for the middle stage and there is no failure expected for the final stage. Advise P&D on this adventure.

PROBLEM 7.6

Pixit & Dindyck are planning a superior real options product PROD that will indicate optimal timing for perpetual multi-stage projects. This time they estimate that the current value of PROD is 87, costs 90 to make in three stages (10 in the current stage, 30 in the middle stage) has a volatility of 20%, cost volatility is 34%, with a -9% correlation of PROD value and cost. The yield on the PROD is expected to be 2%, with no yield for the investment cost. The failure rate of the current stage is 10%, 5% for the middle stage and there is no failure expected for the final stage. Advise P&D on this venture.

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APPENDIX

Three-Stage Model

The extension of the sequential investment model to the $J = 3$ stage is achieved by replication. The value of the option to invest at the $J = 3$ stage F_3 depends on the project value V, and the investment costs at the $J = 1$, $J = 2$ and $J = 3$ stages, K_1, K_2 and K_3, respectively, so $F_3 = F_3(V, K_1, K_2, K_3)$. Using Ito's lemma, it can be shown that the risk neutral valuation relationship for F_3 is:

$$F_3 = A_3 V^{\beta_3} K_1^{\eta_{13}} K_2^{\eta_{23}} K_3^{\eta_{33}}, \tag{7.32}$$

with a simplified characteristic root equation

$$Q_3 = \tfrac{1}{2}\sigma_3^2\phi_3(\phi_3 - 1) + \phi_3\left(r + \lambda_2 - \theta_{K_3}\right) - \left(r + \lambda_3 - \theta_{K_3}\right) = 0. \tag{7.33}$$

$$\begin{aligned} \tfrac{1}{2}\sigma_3^2 = {} & \tfrac{1}{2}\sigma_V^2\phi_2^2\phi_1^2 + \tfrac{1}{2}\sigma_{K_1}^2\phi_2^2(1-\phi_1)^2 + \tfrac{1}{2}\sigma_{K_2}^2(1-\phi_2)^2 + \tfrac{1}{2}\sigma_{K_3}^2 \\ & + \rho_{VK_1}\sigma_V\sigma_{K_1}\phi_1(1-\phi_1)\phi_2^2 + \rho_{VK_2}\sigma_V\sigma_{K_2}\phi_1\phi_2(1-\phi_2)\beta_3\eta_{23} - \rho_{VK_3}\sigma_V\sigma_{K_3}\phi_1\phi_2 \\ & + \rho_{K_1K_2}\sigma_{K_1}\sigma_{K_2}(1-\phi_1)\phi_2(1-\phi_2) - \rho_{K_1K_3}\sigma_{K_1}\sigma_{K_3}(1-\phi_1)\phi_2 - \rho_{K_2K_3}\sigma_{K_2}\sigma_{K_3}(1-\phi_2). \end{aligned} \tag{7.34}$$

$$\hat{V}_3 = \left\{ \frac{\phi_3}{\phi_3 - 1} \frac{\phi_2^{\phi_2}}{(\phi_2 - 1)^{(\phi_2 - 1)}} \left[\frac{\phi_1^{\phi_1}}{(\phi_1 - 1)^{\phi_1 - 1}} \right]^{\phi_2} \right\}^{1/\phi_1\phi_2} \hat{K}_1^{(\phi_1 - 1)/\phi_1} \hat{K}_2^{(\phi_2 - 1)/\phi_1\phi_2} \hat{K}_3^{1/\phi_1\phi_2} \tag{7.35}$$